First order flow only available

Lower limit on the distance-scaled transverse velocity.
Constraint on the immediacy of the surface patch.
Constraint on the immediacy of the plane.
Constraint on the spin.

First order flow and tilt available

The direction of the transverse velocity.
The immediacy of the surface path.
The immediacy of the plane.
The spin.

First order flow and the immediacy of the plane available

The direction of the transverse velocity (4 solutions, coupled to tilt).
The tilt (4 solutions, coupled to transverse velocity).
The immediacy of the surface patch.
The spin (two solutions).

First order flow and spin available

The direction of the transverse velocity (4 solutions, coupled to tilt).
The tilt (4 solutions, coupled to transverse velocity).
The immediacy of the surface patch (2 solutions).
The immediacy of the plane (2 solutions).

It is highly likely that a visual system will have some of this additional information available to it. The ambiguous solutions can be resolved with only coarse extra information.

It is clear that first-order flow, for even a single surface patch, can contribute much of the information needed for the control of actions.

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(43)

(44)

(45)

(46)

These are the components of the deformation rate tensor in the *D*–*S* coordinate system. For the time being, we choose an image coordinate system which has its *x*-axis aligned with *D* and its *y*-axis aligned with *S*. Since we are working in a small patch round the fixed point, where the sphere is close to the tangent plane (Fig. 7), we then have $Q_D = x$ and $Q_S = y$ — that is, etc. We can substitute Eqs. 43 to 46 into Eqs. 31 to 34 to get

(47)

(48)

(49)

(50)

The first three of these are identical to Eqs. 10 to 12. Fig. 4b shows that , so that Eq. 13 is equivalent to Eq. 50.

region about the line of sight the differences will be negligible. All the vectors are defined in a frame of reference which moves with the observer.

In this frame of reference, the surface is translating with a velocity $-\boldsymbol{V}$, and is rotating with an angular velocity $-\omega$. The velocity of the surface point is then given by

$$\frac{\partial}{\partial t}(\rho \boldsymbol{Q}) = -\boldsymbol{V} - \omega \times \rho \boldsymbol{Q} \tag{36}$$

from which it follows (using $\boldsymbol{Q} \cdot \boldsymbol{Q} = 1$) that

$$\boldsymbol{v} = -(\boldsymbol{V} - (\boldsymbol{V} \cdot \boldsymbol{Q})\boldsymbol{Q}) / \rho - \omega \times \boldsymbol{Q} \tag{37}$$

This is the fundamental optic flow equation for rigid body motion.

The equation of the plane

The equation of the plane is simply

$$\boldsymbol{Q} \cdot \boldsymbol{p} = h/\rho \tag{38}$$

where $\boldsymbol{p}$ is the unit vector along the surface normal shown in Fig. 7.

The partial derivatives of flow for a planar surface patch

The next step is to differentiate Eq. 37 with respect to position in the image. To do this, we write it in component form:

$$v_i = (V_j Q_j Q_i - V_i)/\rho - \varepsilon_{ikl}\omega_k Q_l \tag{39}$$

where ε is the alternating tensor and repeated suffices are summed. At this point, the suffices refer to components along arbitrary axes. Differentiation with respect to a component of $\boldsymbol{Q}$ then gives

$$\frac{\partial v_i}{\partial Q_m} = (V_j Q_j Q_i - V_i)\frac{\partial}{\partial Q_m}(1/\rho) + (V_m Q_i + V_j Q_j \delta_{im})/\rho - \varepsilon_{ikm}\omega_k \tag{40}$$

and differentiating both sides of Eq. 38 gives

$$\frac{\partial}{\partial Q_m}(1/\rho) = p_m/h \tag{41}$$

Substituting Eq. 41 into Eq. 40 gives an expression for the first-order flow in arbitrary axes:

$$\frac{\partial v_i}{\partial Q_m} = (V_j Q_j Q_i - V_i)p_m/h + (V_m Q_i + V_j Q_j \delta_{im})/\rho - \varepsilon_{ikm}\omega_k \tag{42}$$

We now simplify this expression by choosing a coordinate frame aligned with the A, D and S axes defined in Fig. 3. We want to evaluate the derivatives at the image of F, where $Q_D = Q_S = 0$ and $Q_A = 1$. We also have $p_S = 0$. From Fig 3b we have $1/\rho = (\sin G)/h$ and from Fig. 3b and Fig. 7 $p_D = -\cos G$. Eq. 42 then becomes

$$R \quad \frac{1}{2} \; v \tag{33}$$

(34)

There are two distinct solutions for θ, at 90˚ to each other, corresponding to the expansion and contraction axes. The expansion axis is given by the solution with and .

The optic flow equation

We now need the equation for the optic flow vector associated with a point on the surface. We denote by ***Q*** a unit vector from *O*, the point of observation, towards some point on the surface. We will write the distance from *O* to the point as ρ, so that the position of the point relative to *O* is given by ρ***Q***. We define the optic flow vector associated with the point to be

(35)

These vectors are shown in Fig. 7. The vector ***Q*** is the position of the image of the surface point onto a spherical image surface of unit radius round *O*, so ***v*** is the velocity vector for that image point. Although most practical image surfaces are planar, in a small

6 Derivation of the first-order flow equations

The equations for first-order optic flow have been derived in a variety of ways by different authors (e.g. Waxman & Wohn, 1988). It is possible to use a variety of formalisms. The following derivation aims to be succinct but reasonably complete and clear.

The Taylor series for optic flow

First, we need to make the connection between the first-order flow variables, as presented in Section 2, and the derivatives of optic flow. It will be convenient to write Eqs. 4 to 6 as a matrix equation:

(28)

or

(29)

where ***v*** is the differential optic flow vector, ***r*** is the position vector in the image, and **T**, the product of the three 2×2 matrices shown in Eq. 28, is the first-order flow deformation tensor.

We can express any optic flow field ***v(r)*** as a Taylor expansion about the origin. This looks like:

(30)

where the first vector on the right is the flow at the origin. We have taken this to be nulled by tracking. We ignore the nonlinear last term on the assumption (that can be justified) that it will be small for low-curvature surfaces and small surface patches. Comparing Eq.

which gives us two possible values for ϕ, one equal to minus the other. The immediacy of F is specified uniquely, from Eq. 22, but there are now two possible values for spin, given by $R \pm S \sin\phi$. There are four possible directions for the transverse velocity: from Eqs. 13 and 20 we find $\angle \boldsymbol{V_T} = \theta - \phi/2$, but as either of the two directions along the expansion axis is possible, and we have uncertainty in the sign of ϕ, we can only state that

$$\angle \boldsymbol{V_T} = \theta + \begin{matrix} 0 \\ \pi \end{matrix} + \begin{matrix} \phi/2 \\ -\phi/2 \end{matrix} \qquad (25)$$

where ϕ in the equation represents one of the possible values, the quantities stacked vertically are alternatives, and all four combinations are allowed. (The angle π is 180˚.) There are four corresponding directions for the tilt, with $+\phi/2$ replaced by $-\phi/2$ and *vice versa*. The possible directions for tilt and transverse velocity given the perpendicular velocity are shown in Fig. 6a.

The ambiguity may well be resolved by other information, which need only be approximate. The spin might be well enough controlled that only one of the two values for ϕ is acceptable, given R, or there might be enough information about the tilt or the direction of the transverse velocity to rule out some of the possible combinations in Eq. 25.

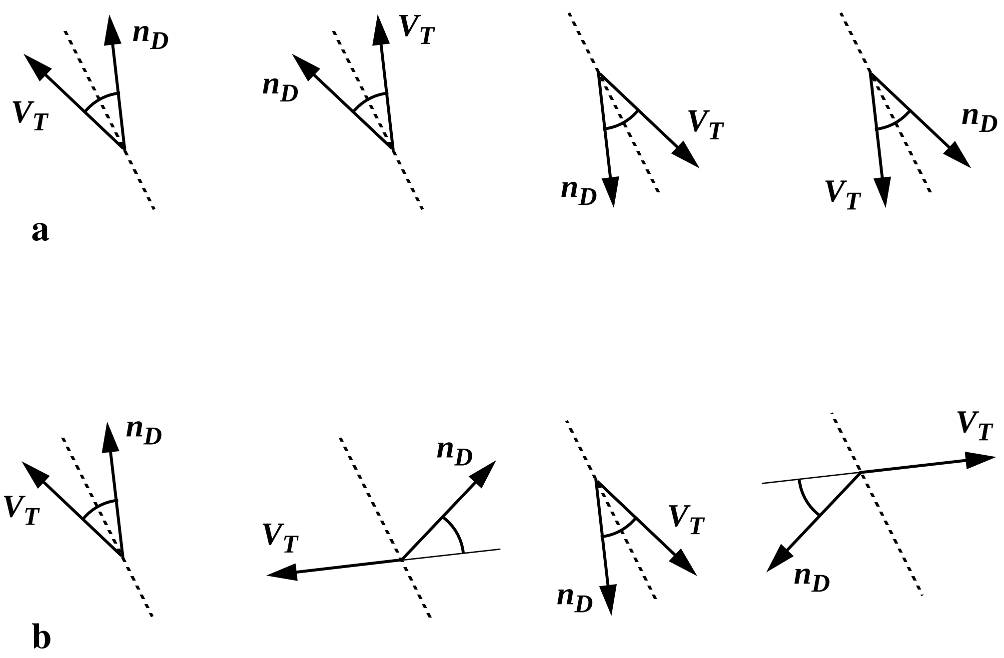


Figure 6. The fourfold ambiguities in tilt and transverse velocity direction. The dotted line is the shear expansion axis and the marked angle shows the value found for ϕ assuming $|\phi| < \pi/2$. (**a**) Perpendicular velocity V_P is known. (**b**) Spin ω_A is known.

Given tilt

It may be the case that we can estimate the direction of the projection of the surface normal, $\angle \boldsymbol{n_D}$. This is called the ***tilt*** of the surface patch. For example, it might be known from the texture gradient, from the direction of the image of a linear object perpendicular to the surface, or from mechanical information. When the surface is a horizontal ground surface, the last two possibilities are very plausible: the observer may be able to see a vertical object or detect the direction of gravity. Using the axis of expansion, we can then deduce the direction of the transverse velocity, and hence the immediacy of F and the spin.

That is, from Eq. 13

$$\angle \boldsymbol{V_T} = 2\theta - \angle \boldsymbol{n_D} \tag{19}$$

and from this and Fig. 4b

$$\phi = \angle \boldsymbol{V_T} - \angle \boldsymbol{n_D} = 2(\theta - \angle \boldsymbol{n_D}) \tag{20}$$

We know θ from the flow and $\angle \boldsymbol{n_D}$ from other information, so we know $\angle \boldsymbol{V_T}$ and ϕ. (The equations refer only to 2θ, so it does not matter that we arbitrarily chose one end of the expansion axis to define θ in Section 2.) Note that ϕ can be negative. From Fig. 4b

$$V_D = V_T \cos\phi \qquad V_S = V_T \sin\phi \tag{21}$$

so from Eqs. 10 to 12

$$1/\tau_F = D - S\cos\phi \tag{22}$$

$$\omega_A = R - S\sin\phi \tag{23}$$

Using Eq. 17 we can also find the immediacy of the plane:

$$1/\tau_P = D - 3S\cos\phi \tag{24}$$

We now have available a good deal of information relevant to the control of locomotion. Even though we do not know the slant, we have the times to nearest approach both to the fixated point and to the plane of the surface, and we know the direction of the projection of our velocity onto the image, so we can predict our future course relative to visible features. In fact, we know everything about our motion and the surface apart from a scale factor in depth.

If we happen to know the surface slant, perhaps from the texture gradient, or because the surface is the ground plane and we know where the horizon is, then we can go further and determine the height-scaled dip, strike and approach velocities. It is by no means certain, however, that these are necessary for control of many actions: the immediacies and the direction of V_T provide a great deal of predictive information as they stand.

Given plane immediacy

If the fixated point is on the ground surface, then plane immediacy will often be mechanically specified. Indeed, V_P will normally be zero. From Eq. 24 we can then obtain cos ϕ,

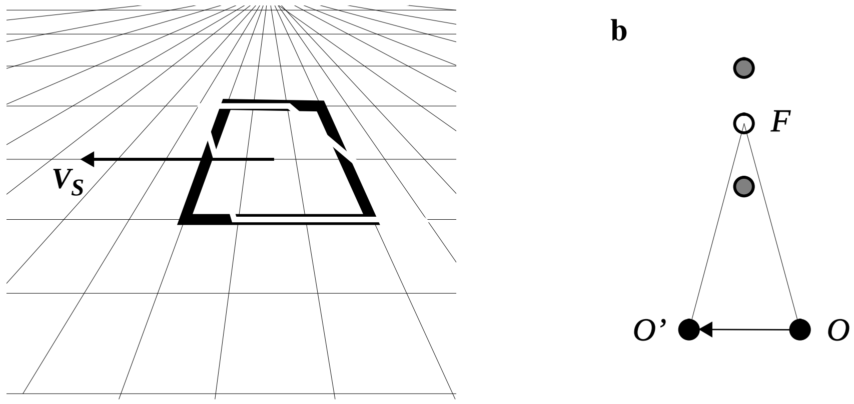


Figure 5. The effect of strike motion. (**a**) The image for an observer moving from right to left and fixating the centre of the outlined rectangle. The shape of the image patch will deform approximately to the shape shown by the dotted outline, so that the flow resembles that of Fig. 1e. There is stretching along the top-left to bottom-right direction and contraction along the bottom-left to top-right direction, and an overall anticlockwise rotation. (**b**) The way this flow arises can be partly understood by looking at a plan view of the scene. As the observer moves from *O* to *O'*, point *A* moves from the left of the line of sight

From Eq. 9, . It follows from Eqs. 10 and 11 that

(15)

That is, we have a constraint on the immediacy of *F*, and if *S* happens to be small compared to *D*, we will have a good estimate of it. Shear is small compared to dilations whenever the slant is small or the transverse velocity is small compared to the approach velocity.

Similarly:

(16)

so there is a limit on the spin.

We can also find a constraint on the immediacy of the plane. This is the inverse of the estimated time to pass through the plane tangent to the surface patch — the ground plane in the examples (and is quite different from the immediacy of *F*). It is equal to the *perpendicular* component of velocity V_P, divided by *h*: we have . From Fig. 3b:

(17)

and it follows from Eqs. 10 and Eqs. 11 that

(18)

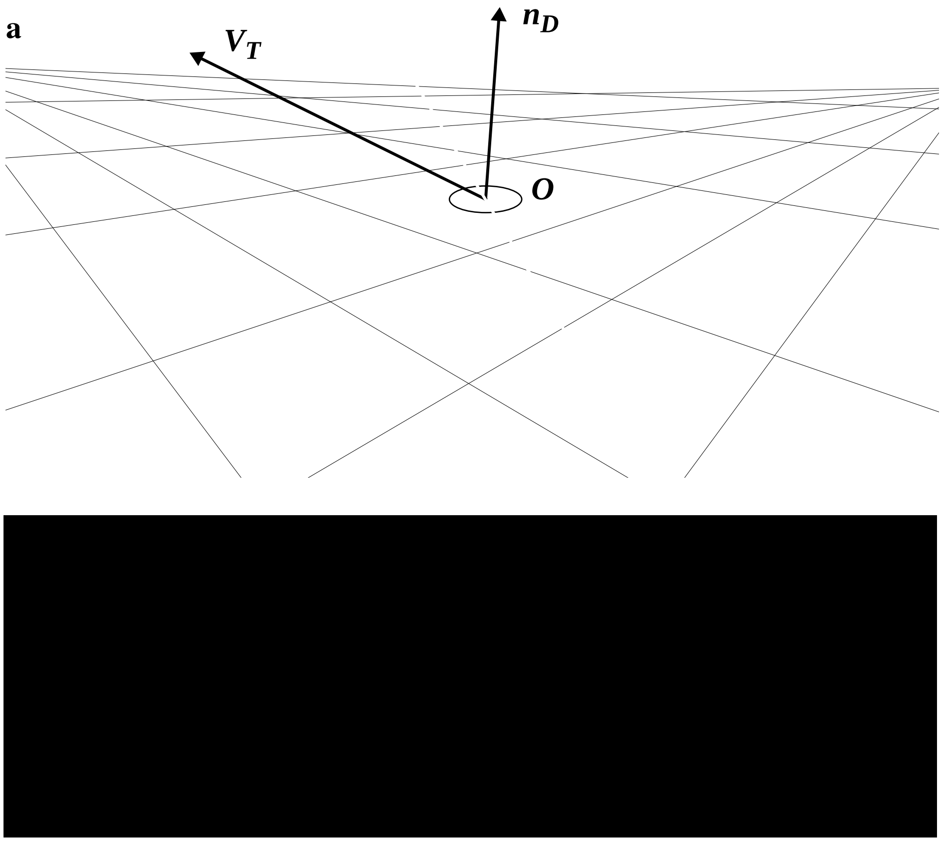


Figure 4. Relationships involving the axis of expansion. (**a**) The figure represents part of the image plane, seen from behind.The axis of expansion is shown by the dotted line, which bisects the angle between $\boldsymbol{n_D}$ and $\boldsymbol{V_T}$. Both of these vectors lie in the plane of the image. Note that $\boldsymbol{n_D}$ is in the direction of the image of a vertical rod with its foot at $\boldsymbol{F}$, and $\boldsymbol{V_T}$ is in the direction of the image of the future path of the observer. (**b**) The same diagram with dip and strike components of velocity shown, and the bisected angle ϕ marked.

to obtain expressions for G and the velocity components in terms of the flow components. In particular, note that V_S and V_D always appear in combination with cos G, so a change in the slant can always be exactly balanced by a change in the transverse velocity, as far as first-order flow is concerned.

Nonetheless, first-order flow clearly carries a great deal of information. Even on its own, the flow for a single patch places strong constraints on the surface slant and tilt and the observer's motion. If some additional information is available, constraining one or more of the quantities involved, a great deal can be inferred. We now explore these possibilities.

No extra information

Since cos G lies between 0 and +1, we have a lower limit on the transverse velocity from Eq. 11:

(14)

4 First-order flow from structure and motion

The equations linking first-order flow to the orientation of the surface and the observer's motion are quite simple. We assume that the image surface is perpendicular to the line of sight, that it is roughly planar close to the fixed point, and that the image is formed by an ordinary optical system approximating to a pinhole, but without inversion. We then have:

(10)

or (11)

(12)

(13)

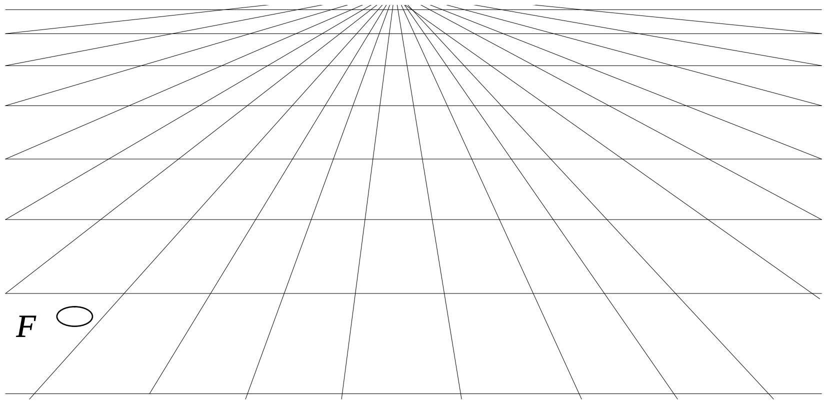
In Eq. 13, $\angle \boldsymbol{n}$ means the angle between the x-axis and the vector $\boldsymbol{n}$, measured anticlockwise from the x-axis. A derivation of the equations is deferred until Section 6.

Let us examine the form of these equations. First, note that all the velocities are divided by $\boldsymbol{h}$, so that $\boldsymbol{h}$ provides, in effect, a distance scale for measuring speed. The dimension of every term in Eqs. 10 to 12 is inverse time.

We can see from Fig. 3b that is equal to , or $1/\tau_F$, where τ_F is the "time-to-contact" — in fact, the time to nearest approach to $\boldsymbol{F}$ if the observer keeps moving with its current velocity. We will call $1/\tau_F$ the ***immediacy*** of F. We can thus say:

where the strike and dip terms include the effect of cos $\boldsymbol{G}$, the slant of the surface at F. The relationship for θ is elaborated in Fig. 4.

The immediacy contribution to the dilation is simply an expression of the fact that the image of an approaching object grows. The dip contribution expresses the stretching that occurs as a result of changing foreshortening: as the observer moves along the $\boldsymbol{D}$ axis towards the surface normal through $\boldsymbol{F}$



example, if you fixate a point on the ground just to the right of your feet as you walk along, your head and eyes will rotate clockwise about your line of sight. We denote the rate of rotation about the line of sight ω_A, and refer to it as the *spin*. It is positive for a clockwise rotation of the eye, looking from behind.

We are now in a position to state the central relationships.

3 Approach, dip and strike

We now return to an observer moving relative to a planar surface, and fixating a point on it. To make the example more concrete, we will take the surface to be the ground surface. The optic flow depends on the instantaneous velocity of the observer's eye, $\boldsymbol{V}$. There are various ways to represent this vector, but it will be convenient to use its components along the A, D and S axes shown in Fig. 3. We make no use at this stage of the fact that motion is normally parallel to the ground plane, as we want the theory to apply to any surface.

The components of velocity along these axes are designated V_A, V_D and V_s, and will be referred to as the where R is the rate of rotation. The formulae are illustrated in Fig. 2c. As with dilation, these equations remain true however the axes are oriented. Positive and negative values of R correspond to anticlockwise and clockwise rotation respectively.

A patch dragged along by a pure rotation clearly suffers no change in shape or in area. A combination of equal parts of rotation and shear produces the kind of parallel or lamellar flow pattern shown in Fig. 1e.

General first-order flow

General first-order flow is formed by simply adding the velocities from the three components. This gives

(4)

However, we will not normally be able to work in a coordinate system conveniently

The geometrical interpretation of these equations is given in Fig. 2a. They hold good regardless of how the axes are oriented. The dilation rate *D* can be positive or negative, corresponding to expansion or contraction respectively.

A patch of the image dragged along with a pure dilational flow suffers no change in shape, but its area changes at a rate of $2DA$, where *A* is its current area.

Shear

Pure shear, or deformation, is the simplest kind of local change of shape. It involves an expansion along some axis and an equal contraction along the axis at right-angles. An example is shown in Fig. 1b.

To write down the equations for shear, assume that the *p*- and *q*-axes of our coordinate system lie along the expansion and contraction axes of the shear respectively. The equations for the velocity are

(2)

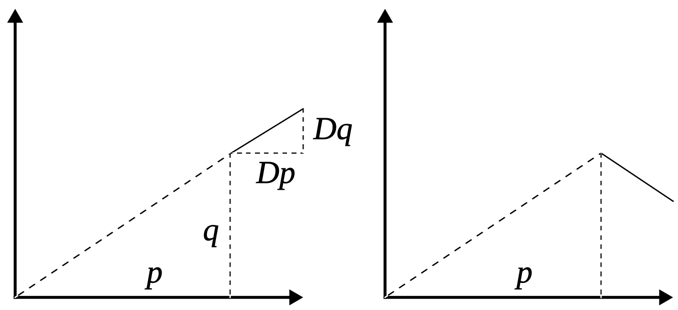
where *S* is the rate of shear, and is always positive. The equations are illustrated in Fig. 2b.

A patch of image dragged along with a pure shear suffers no change in area, but is squeezed along one axis and stretched along another. A combination of equal rates of dilation and shear produces an expansion or contraction in one direction, and no change in the orthogonal direction, as in Fig. 1c.

Rotation

Rotation is just what one would expect. An example is shown in Fig. 1d. Each image point moves at right angles to the line joining it to the fixed point. The equations are

(3)



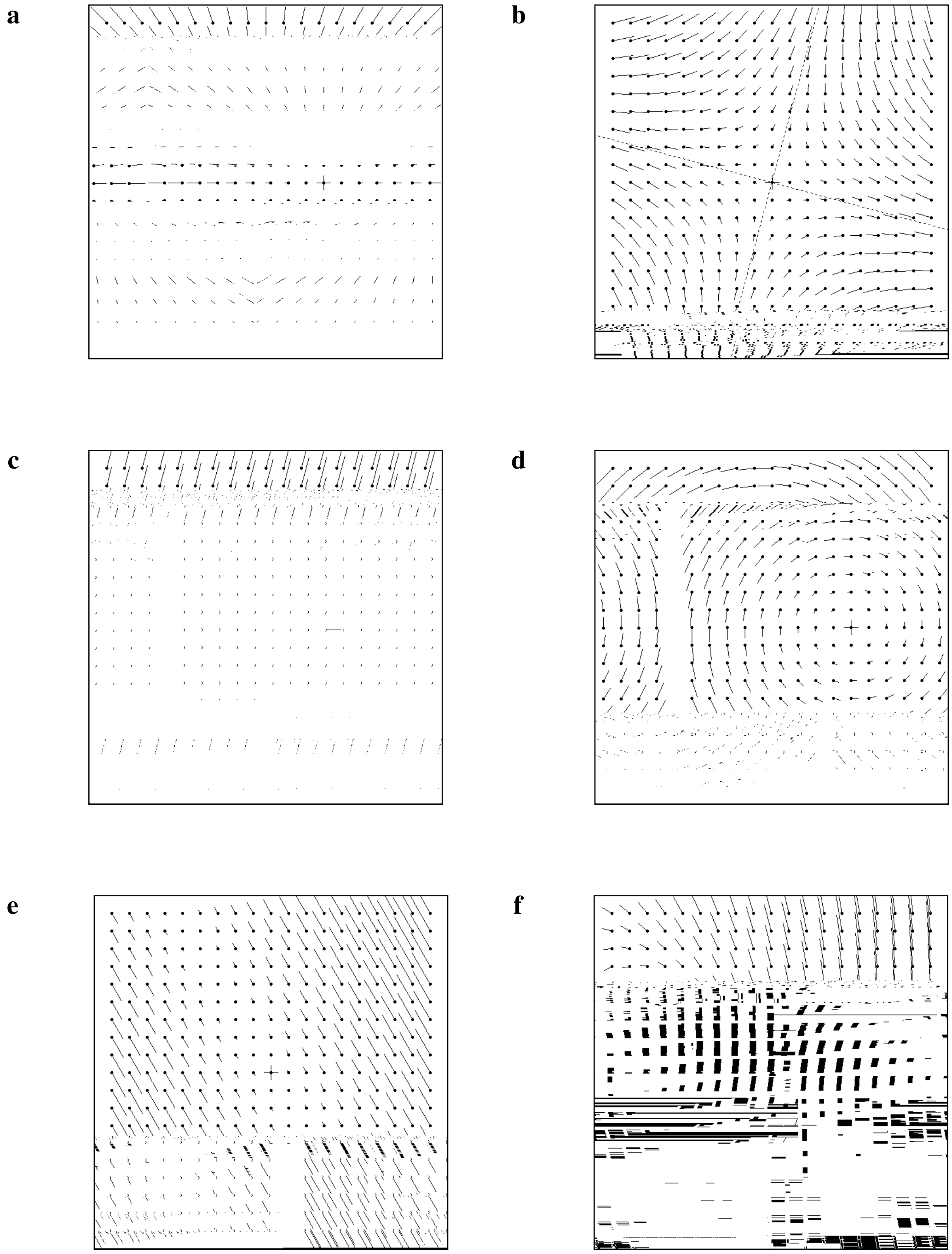


Figure 1. Examples of first-order flow fields. The image velocity for each point marked by a dot is shown by the length and direction of the attached line. The fixed point is shown by a cross. **(a)** Pure dilation. **(b)** Pure shear. The axes of expansion and contraction are shown by the dotted lines. **(c)** A combination of equal rates of dilation and shear. **(d)** Pure rotation. **(e)** A combination of equal rates of rotation and shear. **(f)** A general flow: the combination of a, b and d.

1 Introduction

Walking or driving down a street, you fixate a point on some nearby surface — the pavement or a wall, say. The image of the fixated point is kept static on your retina, but the image of the patch of surface round it changes as you move along, expanding or contracting, deforming, and rotating. This change is the differential optic flow, and it carries information about your motion and about the slant, tilt and shape of the patch of wall or pavement that you are looking at. For example, uniform expansion of the patch is related

First-Order Optic Flow

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